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Letter to the Editor

Dynamic response of a regenerative system to modulated excitation

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1. Introduction

The motivation for the present paper arises from the need to further understand experimental results obtained during wood cutting tests using industrial bandmills. The experimental results show two features of particular interest:

(1) Vibration instability occurs within a narrow speed region at a frequency slightly greater than the natural frequency of the excited mode [1–4]. The experimental results are not well predicted by an undamped stability analysis of the regenerative forces caused by the cutting [4–6].

(2) The vibration response of the blade is characterized by the presence of three frequency peaks: one at the tooth passing frequency, one above and one below with the frequency difference between these peaks corresponding closely to the frequency at which the saw was rotating [2,6]. The dominant vibration response is due to the excitation component at the lowest frequency.

In order to investigate (1), regenerative damping was introduced into the model, and the results of this investigation were reported in Ref. [7]. In order to investigate (2), a modulated forcing function is introduced into the analysis and the present paper discusses the response characteristics that arise. The purpose of this work is to investigate the effect of system variables on the forced vibration response. In the context of saw vibrations the results obtained will assist in the determination of the influence of different parameters upon the vibration characteristics of the saw.

Unwanted vibrations in material machining processing, such as chatter in machine tools [8–10] and washboarding in wood sawing [1–4] are often caused by regenerative cutting forces. Active vibration control systems may also suffer from instability problems due to unavoidable time delayed forces [11–13]. The general class of mathematical problems into which the present problem falls, that of differential difference equations, has an extensive mathematical literature [14–16]. Specific engineering studies on the dynamic stability of a single-degree-of-freedom system (s.d.o.f.) subjected to time-delayed forces have been studied in a number of papers [12,13]. The

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present paper discusses the characteristics of the forced response of such a self-excited system due to a modulated excitation, including the influences of the regenerative force, the modulating frequency, the system damping and the regenerative damping.

2. A regenerative system subjected to modulated force

Fig. 1 shows a s.d.o.f. system with mass m, system damping coefficient c and spring constant k, subjected to a regenerative stiffness force F_r and a regenerative damping force F_d defined by

$$F_r = -k_1[x(t) - x(t-T)], \quad F_d = -c_1[\dot{x}(t) - \dot{x}(t-T)], \tag{1,2}$$

where k_1 is the regenerative stiffness force coefficient, c_1 is the regenerative damping coefficient, and $x(t)$ is the displacement. T is a system time delay, which is assumed to be constant. In a machining process, this time delay corresponds to the period of tooth passage of a cutter.

During milling or sawing operations, the work piece experiences a number of periodic cutting forces. Some of these forces fluctuate at the tooth passing frequency and may be modulated by forces that result from lower-frequency characteristics of the tool. In this work, it will be assumed that the main applied force fluctuates at a frequency ω_t and its amplitude varies with time at a frequency ω_r, called the modulating frequency, which is assumed to be much lower than the original excitation frequency. Thus, the applied force in this model is given by

$$f_m(t) = A_0(1 + a_m \cos \omega_r t) \cos \omega_t t = \sum_{i=1}^{3} F_{mi} \cos \omega_i t, \tag{3}$$

where

$$F_{m1} = A_0, \quad F_{m2} = F_{m3} = A_0 a_m, \tag{4}$$

$$\omega_i = \omega_t(1 - \delta_i) \quad \text{or } \omega = \omega_t(1 - \delta), \tag{5}$$

$$\delta_1 = 0, \quad \delta_2 = -p, \quad \delta_3 = p \quad \text{and} \quad p = \omega_r/\omega_t. \tag{6}$$

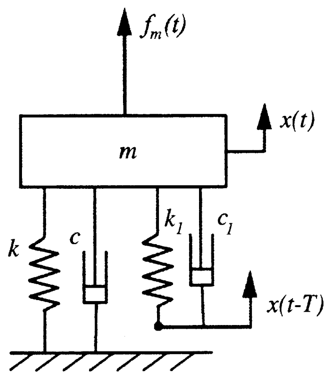

Fig. 1. A s.d.o.f. system subjected to regenerative forces and a modulated external force whose primary excitation frequency is the reciprocal of the time delay of the regenerative forces.

In the above equations, A_0 is the amplitude of the unmodulated force function, a_m the amplitude of the modulation function, ω_r the modulating frequency, and ω_t is the primary excitation frequency, $\omega_t = 2\pi/T$ (T is the time delay); δ is the modulating frequency factor and p is the modulating frequency ratio. It is seen that the applied force $f_m(t)$ contains three components at the frequencies $\omega_t(1-p)$, ω_t and $\omega_t(1-p)$, respectively.

The system is governed by the equation of motion

$$\ddot{x}(t) + 2\zeta_d\omega_n[\dot{x}(t) - r_d\dot{x}(t-T)] + \omega_n^2[(1+r_k)x(t) - r_k x(t-T)] = f(t), \tag{7}$$

where $\omega_n = \sqrt{k/m}$, and $r_k = k_1/k$ is the stiffness ratio. ζ_d is the sum of the system damping ratio ζ and the regenerative damping ratio ζ_c, i.e.,

$$\zeta_d = \zeta + \zeta_c = \frac{c}{2m\omega_n} + \frac{c_1}{2m\omega_n}; \tag{8}$$

$r_d = \zeta_c/\zeta_d$ is defined as the relative damping ratio; $f(t) = f_m(t)/m$ is the modulated excitation.

3. Response to modulated excitation

The homogeneous solution of Eq. (7) governs the system stability characteristics that have been discussed in a companion paper by the authors [7]. The instability regions of such a self-excited system have been determined by considering the effects of the system damping, regenerative damping and regenerative force. In this paper attention will be confined to the response of the system to the specific forcing function $f(t)$.

The particular solution of Eq. (7) is the sum of the solutions to the three force components given by Eq. (3). The steady state response of the system to each excitation component can be expressed in the form

$$x_i(t) = A_i \cos\omega_i t + B_i \sin\omega_i t, \tag{9}$$

where A_i and B_i are constants, $i = 1$, 2 and 3.

The magnitude of the response to the applied forces is given by

$$|x_i(t)| = X_i(\omega)F_{mi}/k, \tag{10}$$

where the magnification factor $X_i(\omega)$ for each excitation component is

$$X_i(\omega) = \frac{1}{\sqrt{Z_i(r_t, \delta_i, r_k, \zeta, r_d)}}, \tag{11}$$

$$\begin{aligned} Z_i(r_t, \delta_i, r_k, \zeta, r_d) = {} & [1 + r_k - r_k\cos 2\pi\delta_i - r_{ti}^2 - 2\zeta_c r_{ti}\sin 2\pi\delta_i]^2 \\ & + [r_k \sin 2\pi\delta_i + 2\zeta_d r_{ti}(1 - r_d\cos 2\pi\delta_i)]^2, \end{aligned} \tag{12}$$

$$r_{ti} = (1+\delta_i)r_t \quad \text{and} \quad r_t = \omega_t/\omega_n, \tag{13}$$

where r_t is the excitation frequency ratio. The response to this modulated excitation can be then expressed by

$$x(t) = \sum_{i=1}^{3} \frac{F_{mi}}{k} X_i(\omega_i)\cos(\omega_i t + \phi_i), \quad i = 1, 2 \text{ and } 3. \tag{14}$$

It is seen that the vibration amplitude is proportional to the force amplitude A_0, the modulation factor a_m and the magnification factors $X_i(\omega)$. It can be seen in the following discussion that the magnification factors of this system are significantly different from those of a conventional system without a regenerative force.

3.1. Response of the system without regenerative damping ($\zeta_c = 0$)

Three magnification factors corresponding to three excitation components at the frequencies $\omega_i = \omega_t(1 + \delta_i)$, $i = 1, 2$ and 3, are shown in Fig. 2. The peak magnification factor for each frequency component can be obtained by maximizing the magnification factor $X_i(\omega)$ defined by Eq. (11). For the given parameters δ, r_k, and ζ, the excitation frequency ratio r_t, at which the value of the function Z is minimized can be found numerically from the condition $\partial Z/\partial r_t = 0$, i.e.,

$$r_{ti}[r_{ti}^2 - 1 - r_k(1 - \cos 2\pi\delta)] + \zeta[r_k \sin 2\pi\delta + 2\zeta r_{ti}] = 0. \tag{15}$$

If the stiffness ratio $r_k \ll 1/(\zeta \sin 2\pi p)$, the solutions for this equation for $\delta = 0$ and $\pm p$ are given by

$$r_{ti} = \frac{1}{1 + \delta_i}\sqrt{1 - 2\zeta^2} \quad \text{or } \omega_{ti} = \frac{\omega_n}{1 + \delta_i}\sqrt{1 - 2\zeta^2}. \tag{16}$$

The peak response occurs when

$$Z_{i\,min}(\delta_i, r_k, \zeta) = [r_k(1 - \cos 2\pi\delta_i) + 2\zeta^2]^2 + \left(2\zeta\sqrt{1 - 2\zeta^2} + r_k \sin 2\pi\delta_i\right)^2, \tag{17}$$

or

$$Z_{1min}(0, r_k, \zeta) = 4\zeta^2(1 - \zeta^2) \approx 4\zeta^2, \tag{18}$$

$$Z_{2min}(-p, r_k, \zeta) = [r_k(1 - \cos 2\pi p) + 2\zeta^2]^2 + \left(2\zeta\sqrt{1 - 2\zeta^2} - r_k \sin 2\pi p\right)^2, \tag{19}$$

$$Z_{3min}(p, r_k, \zeta) = [r_k(1 - \cos 2\pi p) + 2\zeta^2]^2 + \left(2\zeta\sqrt{1 - 2\zeta^2} + r_k \sin 2\pi p\right)^2. \tag{20}$$

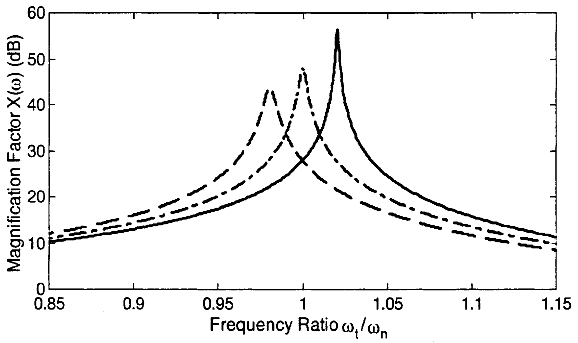

Fig. 2. Effect of primary excitation frequency on magnification factor for the given parameters, $\zeta = 0.002$, $\zeta_c = 0$, $r_k = 0.02$ and $p = 0.02$, and different modulating factors: $- \cdot -\cdot$, $\delta = 0$; ——, $\delta = -0.02$; – – – –, $\delta = 0.02$.

The relationship between the three minimized functions, corresponding to the three different modulating frequency factors, δ_i, satisfies the inequalities

$$Z_{2min}(-p, r_k, \zeta) \leqslant Z_{1min}(0, r_k, \zeta) < Z_{3min}(p, r_k, \zeta) \tag{21}$$

provided that

$$r_k = \frac{k_1}{k} \leqslant r_{k0}, \quad \text{where } r_{k0} = \frac{2\zeta}{\tan(\pi p)} \approx \frac{2\zeta}{\pi p} \tag{22, 23}$$

is referred to as the critical stiffness ratio. If the regenerative stiffness force coefficient k_1 is smaller than the critical value $r_{k0}k$, the peak magnification factor $X_2(\omega)$ for the frequency variation $\delta = -p$ is largest and the dominant vibration response is due to the excitation component at the frequency $\omega_t(1-p)$. Consequently, the following discussion will focus on the term Z_{2min} corresponding to the peak magnification factor.

Fig. 3 shows the excitation frequency ratio at which the peak magnification factor occurs for different modulating frequency factors as a function of the stiffness ratio. It is seen that the ratios of the resonant frequencies to the resonant frequency of the non-regenerative damped system are approximately equal to $1/(1+\delta_i)$ if the stiffness ratio $r_k < 0.1$. Above a stiffness ratio of 1 the excitation frequency ratio corresponding to the peak response increases rapidly.

The corresponding peak magnification factors, as shown in Fig. 4, vary significantly with the stiffness ratio $r_k = k_1/k$ and the modulating frequency factor δ. The magnification factor has a significant peak over a certain range of stiffness ratios when $\delta < 0$. The peak magnification factor of the excitation component at the frequency $\omega_t(1-p)$ reaches its maximum value if the stiffness ratio satisfies $\partial Z_{2min}(-p,\ r_k,\ \zeta)/\partial r_k = 0$. This condition leads to

$$r_{k1} = \frac{k_1}{k} = \frac{\zeta}{\tan(\pi p)} \approx \frac{\zeta}{\pi p} \tag{24}$$

that corresponds to one-half of the critical stiffness ratio.

The peak response component to the excitation component at the frequency $\omega_t(1-p)$ is larger than the components at the frequencies ω_t and $\omega_t(1+p)$ if the stiffness ratio satisfies inequality (22). For the excitation component at the frequency ω_t, the regenerative force is zero and

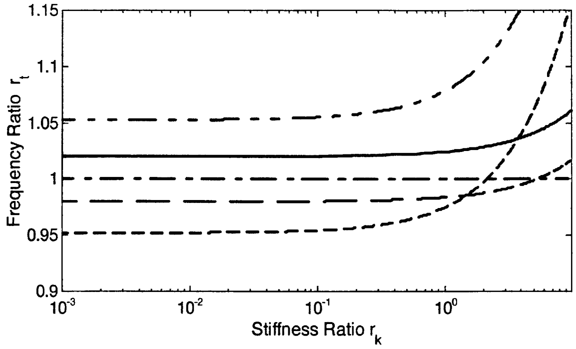

Fig. 3. Effect of stiffness ratio on the excitation frequency ratio corresponding to the peak magnification factor for the given parameters, $\zeta = 0.002$ and $\zeta_c = 0$, and different modulating frequency factors: $- \cdot -$·, $\delta = 0$; ——, $\delta = -0.02$; $- \cdot \cdot -$·, $\delta = -0.05$; — —, $\delta = 0.02$; $- - -$, $\delta = 0.05$.

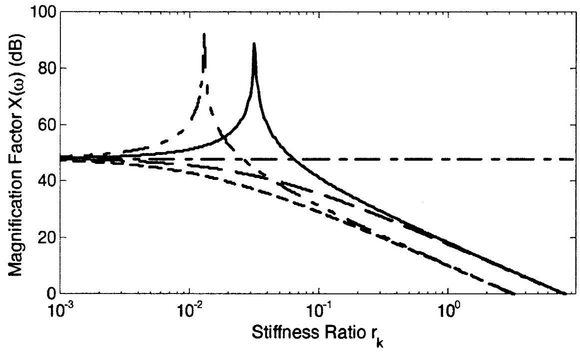

Fig. 4. Variation of magnification factor with the stiffness ratio for the damping ratios, $\zeta = 0.002$ and $\zeta_c = 0$, and different modulating frequency factors: $- \cdot -\cdot$, $\delta = 0$; ——, $\delta = -0.02$; $- \cdot \ \cdot -\cdot$, $\delta = -0.05$; — — , $\delta = 0.02$; $- - -$, $\delta = 0.05$.

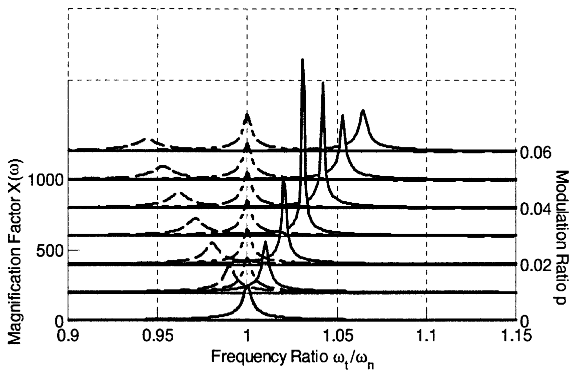

Fig. 5. Variation of magnification factor with excited frequency ratio for the given parameters, $\zeta = 0.002, \zeta_c = 0$ and $r_k = 0.02$, and different modulation factors: $- \cdot -\cdot$, $\delta = 0$; ——, $\delta = -p$; $- - - -$, $\delta = p$.

therefore does not affect the magnification factor. For the component of frequency $\omega_t(1+p)$, the regenerative force always provides an equivalent positive damping and an extra stiffness for this system. For the excitation at the frequency $\omega_t(1-p)$, the regenerative force gives rise to negative "damping" when the stiffness ratio satisfies inequality (22). It gives rise to positive "damping" when the force coefficient is greater than this critical value. If r_k does not satisfy inequality (22), the resonant frequency for $\delta \neq 0$ increases gradually, as shown in Fig. 3, and the magnification factor decreases rapidly (Fig. 4). In this case, the dominant response is that due to the excitation component at the frequency ω_t.

Fig. 5 shows how the magnification factors vary with the excitation frequency for a given range of the parameter p. As expected, the peak magnification factor for $\delta = 0$ does not change since there is no regenerative effect at this frequency. The peak magnification factor for $\delta > 0$ decreases slightly. In this case, if $p \leqslant 0.01$ or $p \geqslant 0.06$, the peak magnification factor for $\delta = -p$ is small. When $p = \zeta k/(\pi k_1)$ corresponding to one-half of the critical stiffness ratio the magnification factor achieves a maximum value.

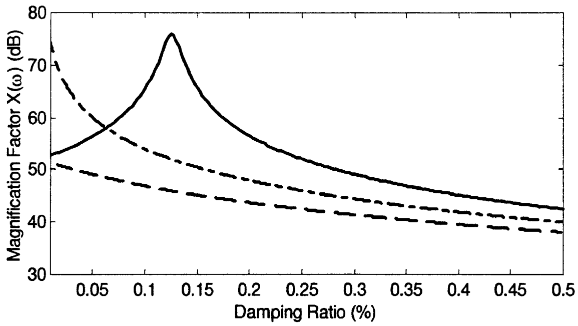

Fig. 6. Variation of peak magnification factors with the system damping ratio for the given parameters, $r_k = 0.02$ and $\zeta_c = 0$, and different modulating frequency factors: $- \cdot -\cdot$, $\delta = 0$; ——, $\delta = -0.02$; – – – –, $\delta = 0.02$.

The system damping affects three peak magnification factors. Fig. 6 shows how the three peak magnification factors vary with the system damping ratio. The magnification factor corresponding to the excitations for $\delta = 0$ decreases with the system damping ratio. When the system damping is very light, the magnification factor corresponding to the excitation at the frequency ω_t $(\delta = 0)$ has the largest values. Its response then dominates the vibration of the system. The magnification factor corresponding to the excitation at $\omega_t(1 + p)$ $(\delta > 0)$ decreases monotonically with increased damping ratio. For a given regenerative force coefficient k_1 and modulating frequency factor p, the peak magnification factor $(\delta = -p)$ is maximized with respect to the system damping ratio ζ if the damping ratio satisfies

$$\frac{\partial Z_{2min}(-p, r_k, \zeta)}{\partial \zeta} = 0. \tag{25}$$

Assuming that the modulation frequency is much lower than the original excitation frequency, i.e., $p \ll 1$, this leads to the result that

$$\zeta_{peak} = \frac{r_k}{2} \sin 2\pi p \approx \pi p r_k = \pi p k_1 / k. \tag{26}$$

If the system damping ratio $\zeta < \pi p k_1 / k$, the effective damping in the system is negative and the magnification factor increases with the damping ratio. When the system damping ratio is greater than this value, the magnification factor decreases monotonically with the damping ratio.

3.2. Response of the system with regenerative damping ($\zeta_c \neq 0$)

If the primary damped system is also subjected to regenerative damping, the steady state response to the modulated excitation will be significantly modified. The frequency at which peak magnification occurs, for given parameters δ, r_k, ζ and ζ_c, can be determined based on the excitation frequency ratio r_{ti} that satisfies $\partial Z / \partial r_{ti} = 0$, i.e.,

$$\begin{aligned} &[r_{ti}^2 + 2\zeta_c r_{ti} \sin 2\pi\delta - 1 - r_k(1 - \cos 2\pi\delta)][r_{ti} + 2\zeta_c \sin 2\pi\delta] \\ &\quad + \zeta_d(1 - r_d \cos 2\pi\delta)[r_k \sin 2\pi\delta + 2\zeta_d r_{ti}(1 - r_d \cos 2\pi\delta)] = 0. \end{aligned} \tag{27}$$

An approximate solution for r_{ti} is found with the assumption, $(r_k\zeta - \zeta_c)\sin 2\pi\delta_i \ll 1$, and given by

$$r_{ti} = \sqrt{1 + R_{ti}} - 1.5\zeta_c \sin 2\pi\delta_i, \tag{28}$$

$$R_{ti} = r_k(1 - \cos 2\pi\delta_i) + (0.5\zeta_c \sin 2\pi\delta_i)^2 - 2\zeta_d^2(1 - r_d \cos 2\pi\delta_i)^2. \tag{29}$$

In the case where $\delta = 0$, $r_{ti} = r_t$ and the peak magnification factor is the same as that of a conventional damped system with no regenerative forces. The maximum magnification factor $X_0(\omega_n) \approx 1/(2\zeta)$, which will be taken as a reference magnification factor for the following discussion. In the case of $\delta = \pm p$, the excitation frequency ratios r_{ti} given by Eq. (28) have very small shifts away from one. For simplicity, the peak magnification factors will be approximated by their values at $r_{ti} = 1$. These peak values are associated with the functions given by

$$\begin{aligned} Z_{i\,min}(\delta_i, r_k, \zeta, \zeta_c) = &[r_k(1 - \cos 2\pi\delta_i) - 2\zeta_c \sin 2\pi\delta_i]^2 \\ &+ [r_k \sin 2\pi\delta_i + 2\zeta_d(1 - r_d \cos 2\pi\delta_i)]^2 \end{aligned} \tag{30}$$

or

$$Z_{1min}(0, r_k, \zeta, \zeta_c) = 4\zeta^2, \tag{31}$$

$$\begin{aligned} Z_{2min}(-p, r_k, \zeta, \zeta_c) = &[r_k(1 - \cos 2\pi p) + 2\zeta_c \sin 2\pi p]^2 \\ &+ [-r_k \sin 2\pi p + 2\zeta_d(1 - r_d \cos 2\pi p)]^2, \end{aligned} \tag{32}$$

$$\begin{aligned} Z_{3min}(p, r_k, \zeta, \zeta_c) = &[r_k(1 - \cos 2\pi p) - 2\zeta_c \sin 2\pi p]^2 \\ &+ [r_k \sin 2\pi p + 2\zeta_d(1 - r_d \cos 2\pi p)]^2. \end{aligned} \tag{33}$$

Fig. 7 illustrates the variation of the peak magnification factors with the stiffness ratio for $\zeta = \zeta_c = 0.002$. The factors for $\delta > 0$ decrease steadily with the stiffness ratio and are always less than the reference magnification factor for $\delta = 0$. The factors for $\delta < 0$ are greater than the reference factor in the ranges $[I_{a1}, I_{b1}]$ for $\delta = -0.02$ and $[I_{a2}, I_{b2}]$ for $\delta = -0.05$. They reach the maximum values at $r_k \approx 0.032$ and $r_k \approx 0.013$, respectively. Compared with the case of $\zeta_c = 0$

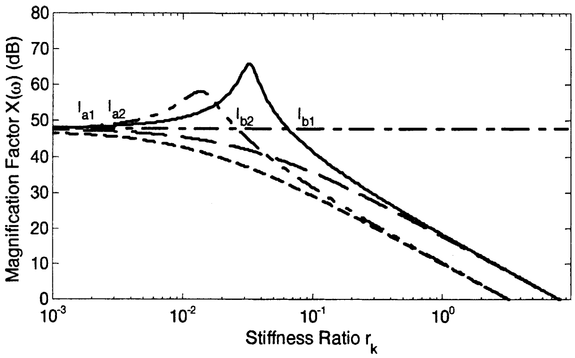

Fig. 7. Variation of magnification factor with the stiffness ratios for the given damping ratios, $\zeta = \zeta_c = 0.002$ and different modulation frequency factors: $- \cdot -\cdot$, $\delta = 0$; ——, $\delta = -0.02$; $- \cdot \cdot -\cdot$, $\delta = -0.05$; — —, $\delta = 0.02$; $- - -$, $\delta = 0.05$.

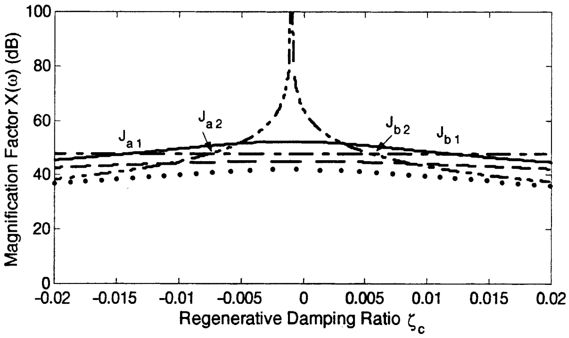

Fig. 8. Variation of magnification factors with the regenerative damping ratio for the given parameters, $\zeta = 0.002$ and $r_k = 0.013$, and different modulation frequency factors: $-\cdot-\cdot$, $\delta = 0$; ——, $\delta = -0.02$; $-\cdot\cdot-$, $\delta = -0.05$; — —, $\delta = 0.02$; ...., $\delta = 0.05$.

shown in Fig. 4, the regenerative damping reduces the peak values, but it does not affect the locations of the peaks.

The variation of the peak magnification factors with the regenerative damping for a given stiffness ratio is shown in Fig. 8. The peak magnification factors greater than the reference factor occur in the range $[J_{a1}, J_{b1}]$ for $\delta = -0.02$ and in the range $[J_{a2}, J_{b2}]$ for $\delta = -0.05$. The peak magnification factor for $\delta = -0.05$ and $r_k = 0.013$ approaches infinity as $\zeta_c \to -0.001$.

It is seen from Eqs. (31) and (33) that if $\zeta_c = 0$, $Z_{3min}(p, r_k, \zeta, 0) > 4\zeta^2$. If $\zeta_c > 0$, $\partial Z_{3min}/\partial \zeta_c > 0$ and Z_{3min} increases with ζ_c. Then it is concluded that

$$Z_{1min}(0, r_k, \zeta, \zeta_c) < Z_{3min}(p, r_k, \zeta, \zeta_c) \tag{34}$$

holds for small positive parameters. The inequality

$$Z_{2min}(-p, r_k, \zeta, \zeta_c) \leqslant Z_{1min}(0, r_k, \zeta, \zeta_c) \tag{35}$$

holds if the parameters r_k and ζ_c are selected such that they lie inside an ellipse described by

$$\frac{(r_k - r_{k1})^2}{(r_{k0}/2)^2} + \frac{(\zeta_c + \zeta/2)^2}{(r_{k0}/4)^2} = 1, \tag{36}$$

where

$$r_{k0} = \frac{2\zeta}{\sin(\pi p)}, \quad r_{k1} = \frac{\zeta}{\tan(\pi p)}. \tag{37}$$

Fig. 9 illustrates two ellipses defined in the domain (r_k, ζ_c) with the center at $(r_{k1}, -\zeta/2)$ for $\delta = -0.02$ and -0.05. The lengths of the major and minor axes of each ellipse are r_{k0} and $r_{k0}/2$, respectively. The solid line indicates that the parameters in this system are positive. The lower half of an ellipse with a dashed line indicates that a negative regenerative damping ratio is used. If the parameters, δ, r_k, ζ and ζ_c, selected lie inside an ellipse, then $Z_{2min} < Z_{1min}$ or the peak magnification factor for $\delta = -p$ is the largest one. If they are exactly on the ellipse, then

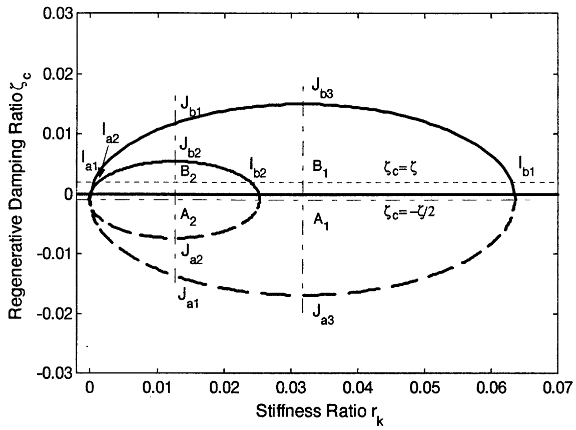

Fig. 9. A bigger and a smaller ellipses defining the variation of parameters for the peak magnification factors for two modulating frequency factors $\delta = -0.02$ and -0.05, respectively.

$Z_{2min} = Z_{1min}$ and the factors for both $\delta = 0$ and $\delta = -p$ are the same. If the parameters lie outside the ellipse, then $Z_{2min} > Z_{1min}$ and the factor for $\delta = -p$ is less than the reference factor. It is noted that the size and the location of the ellipse depend on the system damping ratio ζ and the modulating frequency factor p. A larger system damping ratio or a smaller modulating frequency factor gives rise to a larger ellipse in which the dominant response is caused by the excitation with $\delta = -p$.

For parameters corresponding to points within this ellipse, the peak magnification factor $X_2[\omega_n/(1-p)]$ for $\delta = -p$ can be maximized by minimizing Z_{2min} with respect to the parameters, r_k, ζ_c and p. The results are given by

$$\frac{\partial Z_{2min}}{\partial r_k} = 0 \quad \rightarrow \quad r_k = \frac{\zeta}{\tan \pi p}, \tag{38}$$

$$\frac{\partial Z_{2min}}{\partial \zeta_c} = 0 \quad \rightarrow \quad \zeta_c = -\frac{\zeta}{2}, \tag{39}$$

$$\frac{\partial Z_{2min}}{\partial p} = 0 \quad \rightarrow \quad \frac{(r_k - \bar{r}_{k1})^2}{\bar{r}_{k0}^2} + \frac{(\zeta_c + \zeta/2)^2}{(\bar{r}_{k0}/2)^2} = 1, \tag{40}$$

where

$$\bar{r}_{k0} = \frac{\zeta}{\sin 2\pi p}, \quad \bar{r}_{k1} = \frac{\zeta}{\tan 2\pi p}. \tag{41}$$

It is seen from Eqs. (38)–(40) that the minimized function Z_{2min} for a given system damping ζ and a given modulating frequency factor p occurs at the center (point A_1 or A_2) of an ellipse shown in Fig. 9. In this case, the energy supplied by the regenerative force and the negative regenerative damping completely cancel the energy consumed by the system damping. This system behaves as an undamped system with an infinite magnification factor at the resonant frequency.

For a constant regenerative damping ratio ζ_c, such as $\zeta_c = \zeta = 0.002$, the function Z_{2min} varies with the stiffness ratio r_k along a line passing Points I_{a1}, I_{a2}, I_{b2} and I_{b1} that corresponds to four points shown in Fig. 7. The minimum value of this function occurs at the intersection, Point B_1 or B_2, of this line and the minor axis (J_{a2} J_{b2} or J_{a3} J_{b3}) of the ellipse. If the stiffness ratio r_k is given, the function Z_{2min} varies with the regenerative damping ζ_c along a line, such as J_{a1} J_{b1} or J_{a2} J_{b2} that corresponds four points shown in Fig. 8. The minimum value of this function occurs at $\zeta_c = -\zeta/2$ and the function value increases as the regenerative damping moves away from this point along a line parallel to the axis ζ_c.

The effect of the modulating frequency factor on the peak magnification factor can also been studied using the ellipse. For a given system damping ratio ζ, the size and the location of the ellipse vary with the modulating frequency ratio p. The minimum value of the function Z_{2min} always occurs at the center $(r_{k1}, -\zeta/2)$. If the modulating frequency ratio $p \to 0$, the ellipse size approaches infinity. In this case, the system will be subjected only to an exciting force and the regenerative forces due to the steady state motion will vanish no matter what stiffness ratio and regenerative damping ratio are selected.

4. Conclusions

A study of the forced vibration response of a s.d.o.f. system subjected to regenerative and modulated forces has been conducted. The analytical relationships between the system damping ratio, the regenerative stiffness force, the regenerative damping force, the modulated excitation and the response have been established.

If the amplitude of an applied harmonic force at a frequency equal to the reciprocal of the time delay is modulated by a low-frequency component, the force consists of components at three distinct frequencies. The corresponding resonance peaks occur at the frequencies that are slightly lower, higher than and equal to the resonant frequency of the unmodulated system. If the ratio of the regenerative force coefficient to the system spring constant is small, the maximum peak-magnification factor is caused by the excitation component at the lowest modulated frequency. This interesting behavior is due to the coupling effect of the time delay of the regenerative force and the frequency of the modulated forces.

The effects, on the magnification factors of the system, of four factors: the system damping, the regenerative damping, the regenerative force coefficient and the modulating frequency are closely coupled with each other. The parameters that give rise to the maximum magnification factors have been determined analytically for particular conditions. In general, the peak magnification factor of the higher frequency component is not greater than the peak factor of the unmodulated frequency component. The only factor affecting the peak magnification factor of the unmodulated frequency component is the system damping. It is shown that increasing the system damping decreases the response magnitude. Whether the system damping increases or decreases the magnification factor of the lower-frequency component is dependent on the stiffness ratio, the modulating frequency and the regenerative damping. The coupling effects of these variables have been described geometrically in terms of an ellipse in the plane of the stiffness ratio and the regenerative damping ratio. The peak magnification factor can be determined with respect to these parameters using the geometry of the ellipse.

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